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A large deviation principle for the Brownian snake

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Abstract

We consider the path-valued process called the Brownian snake, conditioned so that its lifetime process is a normalised Brownian excursion. This process denoted by $((W_s, \zeta_s);\ s \in [0, 1])$ is closely related to the integrated super-Brownian excursion studied recently by several authors. We prove a large deviation principle for the law of

$$((\varepsilon W_s(\zeta_s),\ \varepsilon^{2/3}\zeta_s);\ s \in [0, 1])$$

as $\varepsilon \downarrow 0$. In particular, we give an explicit formula for the rate function of this large deviation principle. As an application we recover a result of Dembo and Zeitouni.

Keywords: Brownian snake; Large deviation principle; Super-Brownian motion; Rate function

AMS classification: 60F10; 60G15; 60J25

1. Introduction and statement of results

The Brownian snake is a random process whose values are paths in $\mathbb{R}^d$. This process introduced by Le Gall has given many interesting results concerning super-Brownian motion as well as a nice probabilistic representation of the solutions of the partial differential equation $\Delta u = u^2$ (see Le Gall, 1993, 1994). The Brownian snake is also related to the models proposed by Aldous (1993) as fundamental models for random distribution of mass. In particular, the Brownian snake can be used to construct the so-called integrated super-Brownian excursion. The reader can consult Derbez and Slade (1996) for some recent results on the integrated super-Brownian excursion.

Dembo and Zeitouni (1993a) give a large deviation type result for a model of random distribution of mass. Roughly speaking, they estimate the probability as $\varepsilon \downarrow 0$ that the range of the Brownian snake normalised by ε hits a finite number of fixed balls. However, they do not provide a general large deviation principle. We propose in this paper to continue the study by proving a large deviation principle with an explicit rate

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function. The paper is organised as follows. In the present section we introduce a few notations, recall the definition of the Brownian snake and state our main result. In Section 2 we investigate the finite-dimensional marginal laws of the Brownian snake. We use these results in Section 3 to prove a large deviation principle for the finite-dimensional marginal laws of the Brownian snake. In Section 4 we prove a result of exponential tightness. Section 5 is then devoted to the proof of the main theorem. Finally, Section 6 gives two applications and in particular a new proof of the result of Dembo and Zeitouni (1993a).

Let us first recall the definition of the Brownian snake. For a detailed exposition the reader should refer to Le Gall (1993, 1994). We denote by $C([0, +\infty), \mathbb{R}^d)$ the space of all continuous functions from $[0, +\infty)$ to $\mathbb{R}^d$. We call stopped path in $\mathbb{R}^d$ a pair $(w, \zeta) \in C([0, +\infty), \mathbb{R}^d) \times [0, +\infty)$ such that, for every $t \geq \zeta$, $w(t) = w(\zeta)$. We denote by $\mathscr{W}$ the space of all stopped paths in $\mathbb{R}^d$. We call ζ the lifetime of the stopped path (w, ζ). We denote $\hat{w} = w(\zeta)$ the "terminal point" of the path w. The space $\mathscr{W}$ is a complete metric space when equipped with the metric

$$\operatorname{dist}((w, \zeta); (w', \zeta')) = \sup_{t \geq 0} |w(t) - w'(t)| + |\zeta - \zeta'|.$$

In the present work, we study the Brownian snake whose lifetime is a normalised Brownian excursion and whose initial point is 0. This means that, if $((W_s, \zeta_s); s \in [0, 1])$ denotes such a process under a probability $\boldsymbol{P}_0$, it takes its values in $\mathscr{W}$ and has the following properties:

- for every $s \in [0, 1]$, $W_s(0) = 0$,
- $(\zeta_s; s \in [0, 1])$ has the law of the normalised Brownian excursion (as defined for example in Blumenthal (1992, p. 42) or Revuz and Yor (1994, p. 462).
- the conditional law of (W_s) knowing (ζ_s) is the law of an inhomogeneous Markov process whose transition kernel is described as follows:

for $0 \leq s < t \leq 1$,

- $W_t(u) = W_s(u)$ for all $u \leq m(s, t) = \inf_{s \leq u \leq t} \zeta_u$,
- conditionally given $W_s(m(s, t))$, $(W_t(m(s, t) + u), 0 \leq u \leq \zeta_t - m(s, t))$ is independent of W_s and distributed as a Brownian motion in $\mathbb{R}^d$ starting from $W_s(m(s, t))$ and stopped at time $\zeta_t - m(s, t)$.

As a consequence, under $\boldsymbol{P}_0$, the process $((W_s, \zeta_s); s \in [0, 1])$ takes its values in the space $\mathscr{PW}$ of all $((U_s, \eta_s); s \in [0, 1]) \in C([0, 1], \mathscr{W})$ such that $\eta_0 = \eta_1 = 0$, $U_s(0) = 0$ for every $s \in [0, 1]$ and for all $0 \leq s \leq t \leq 1$, for every $u \leq \inf_{[s, t]} \eta$, $U_t(u) = U_s(u)$.

We say that a function $f: [0, 1] \to \mathbb{R}^m$ is absolutely continuous if there exists an integrable function $\dot{f}: [0, 1] \to \mathbb{R}^m$ such that, for every $t \in [0, 1]$, $f(t) = \int_0^t \dot{f}(s)\, \mathrm{d}s$. In this case, for almost every $s \in [0, 1]$, f is differentiable at point s and its derivative is $\dot{f}(s)$. Absolute continuity of f is equivalent to the fact that, for every $\varepsilon > 0$, there exists $\alpha > 0$ such that, for every integer $n \geq 1$, for all $0 < a_1 < b_1 < \cdots < a_n < b_n < 1$ satisfying $\sum_{i=1}^n |b_i - a_i| < \alpha$, we have $\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon$. We denote by $\mathscr{HPW}$ the space of all $(U, \eta) \in \mathscr{PW}$ such that (η_s) and $(\hat{U}_s)$ are two absolutely continuous function from $[0, 1]$ to $[0, +\infty)$ and $\mathbb{R}^d$, respectively.

Our large deviation result deals with the process $((\varepsilon \hat{W}_s, \varepsilon^{2/3} \zeta_s); s \in [0, 1])$ (where we recall the notation $\hat{W}_s = W_s(\zeta_s)$). If we set $\zeta_s^\varepsilon = \varepsilon^{2/3} \zeta_s$ and $W_s^\varepsilon(u) = \varepsilon W_s(\varepsilon^{-2/3} u)$ we remark that $(\varepsilon \hat{W}_s, \varepsilon^{2/3} \zeta_s) = (\hat{W}_s^\varepsilon, \zeta_s^\varepsilon)$. As a consequence the process $((\varepsilon \hat{W}_s, \varepsilon^{2/3} \zeta_s);$

$s \in [0,1]$) takes its values in the set $\mathscr{E}$ of all $(\hat{U}, \eta)$ with $(U, \eta) \in \mathscr{PW}$. The set $\mathscr{E}$ is endowed with the topology induced on $C([0,1], \mathbb{R}^d) \times C([0,1], [0, +\infty))$ by the norm

$$\|(T, \eta)\| = \sup_{s \in [0,1]} (|\eta_s| + |T_s|).$$

Finally, we denote $\mathscr{HE}$ the space of all $(\hat{U}, \eta)$ with $(U, \eta) \in \mathscr{HPW}$.

Theorem 1. *The laws under P_0 of $((\varepsilon \hat{W}_s, \varepsilon^{2/3}\zeta_s);\ s \in [0,1])$ satisfy a large deviation principle as $\varepsilon \downarrow 0$ with speed $\varepsilon^{-4/3}$ and good rate function J defined by*

$$J(T, \eta) = \frac{1}{2} \int_0^1 \dot{\eta}_s^2 \, ds + \frac{1}{4} \int_0^1 \frac{|\dot{T}_s|^2}{|\dot{\eta}_s|} \, ds$$

if $(T, \eta) \in \mathscr{HE}$ and $+\infty$ if not.

By the previous theorem we mean that

- for every $L \in [0, +\infty)$, the set $J^{-1}([L, +\infty))$ is compact,
- for every open subset U of $\mathscr{E}$,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{4/3} \log P_0 \, [(\varepsilon \hat{W}_s, \varepsilon^{2/3}\zeta_s)_{s \in [0,1]} \in U] \geqslant - \inf_U J$$

- for every closed subset K of $\mathscr{E}$,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{4/3} \log P_0 \, [(\varepsilon \hat{W}_s, \varepsilon^{2/3}\zeta_s)_{s \in [0,1]} \in K] \leqslant - \inf_K J.$$

2. Marginal laws of the Brownian snake

Our first goal is to prove a large deviation principle concerning the finite-dimensional marginals of $(\varepsilon W_s, \varepsilon^{2/3}\zeta_s;\ s \in [0,1])$. This requires a better understanding of the law of the $2n$-tuple $(\hat{W}_{u_1}, \ldots \hat{W}_{u_n}, \zeta_{u_1}, \ldots \zeta_{u_n})$ with $0 < u_1 < \cdots < u_n < 1$. We know that the two stopped paths W_{u_i} and $W_{u_{i+1}}$ coincide up to time $\inf_{[u_i, u_{i+1}]} \zeta$. We note that, under P_0, there exists a unique m_i such that $\zeta_{m_i} = \inf_{[u_i, u_{i+1}]} \zeta$ and we use the notation $m_i = \operatorname{arginf}([u_i, u_{i+1}], \zeta)$. In fact, it is interesting to study the law under P_0 of the (4n − 2)-tuple

$$(\hat{W}_{u_1}, \ldots, \hat{W}_{u_n}, \zeta_{u_1}, \ldots, \zeta_{u_n}, \zeta_{m_1}, \ldots, \zeta_{m_{n-1}}, \hat{W}_{m_1}, \ldots, \hat{W}_{m_{n-1}}).$$

In Proposition 2 we first give the conditional law knowing (ζ_s). This result is taken from Serlet (1995). Then, in Proposition 3, we give the law of the values of (ζ_s) which appear in the previous proposition. We need to recall some notations introduced in Serlet (1995). For a pictorial description see Fig. 1.

Let $\alpha_1, \ldots, \alpha_{n-1}$ be distinct positive real numbers. We denote by $A(\alpha_1, \ldots, \alpha_{n-1})$ the mapping $a: \{1, \ldots, n-1\} \to \{0, \ldots, n-1\}$ given, for every $i \in \{1, \ldots, n-1\}$, by $a(i) = l$ with

$$\alpha_l < \alpha_i, \quad \forall j \in (l \wedge i, l \vee i),\ \alpha_j > \alpha_i,\ \alpha_l \text{ as large as possible},$$

if such an integer l exists and $a(i) = 0$ otherwise. We also define a mapping $v: \{1, \ldots, n\} \to \{1, \ldots, n-1\}$ by setting $v(1) = 1$, $v(n) = n-1$ and, for $i \in \{2, \ldots, n-1\}$ $v(i) = i-1$ if $\alpha_{i-1} > \alpha_i$ and $v(i) = i$ otherwise. As it is proved in

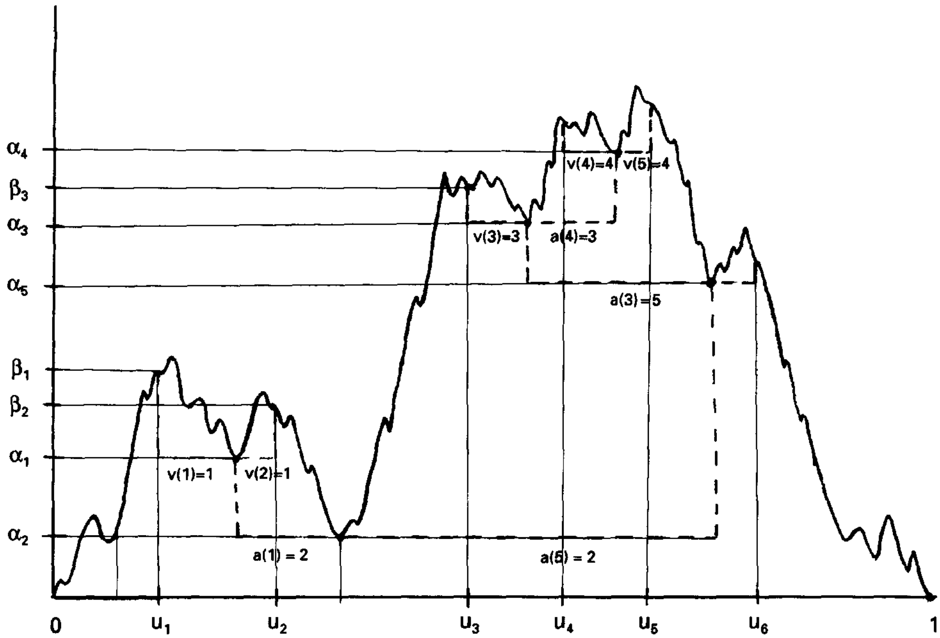

Fig. 1.

Serlet (1995), the mapping v is determined by $a = A(\alpha_1, \dots, \alpha_{n-1})$ so that we use the notation v_a for v. We denote by $p(\cdot\,;\cdot,\cdot)$ the Gaussian transition density in $\mathbb{R}^d$: $p(t;\, x, y) = (2\pi t)^{-d/2}\exp(-|x-y|^2/2t)$. The following proposition is Theorem 6 of Serlet (1995), expressed in a slightly different setting since in Serlet (1995), $\boldsymbol{P}_0$ is the excursion measure of the Brownian snake. But the result is still valid in the present situation. We recall the notation $m_i = \operatorname{arginf}([u_i, u_{i+1}],\ \zeta)$.

Proposition 2. *Under $\boldsymbol{P}_0$, the conditional law of $(\hat{W}_{u_1}, \dots, \hat{W}_{u_n}, \hat{W}_{m_1}, \dots, \hat{W}_{m_{n-1}})$ knowing (ζ_s) has a density with respect to the Lebesgue measure given by the formula*

$$\boldsymbol{P}_0(\hat{W}_{u_1} \in \mathrm{d}y_1, \dots, \hat{W}_{u_n} \in \mathrm{d}y_n, \hat{W}_{m_1} \in \mathrm{d}z_1, \dots, \hat{W}_{m_{n-1}} \in \mathrm{d}z_{n-1}) | (\zeta_s))$$

$$= \prod_{i=1}^{n-1} p(\alpha_i - \alpha_{a(i)};\, z_{a(i)}, z_i) \prod_{i=1}^{n} p(\beta_i - \alpha_{v_a(i)};\, z_{v_a(i)}, y_i)\, \mathrm{d}y_1 \dots \mathrm{d}y_n\, \mathrm{d}z_1 \dots \mathrm{d}z_{n-1},$$

where $\alpha_i = \inf_{s \in [u_i, u_{i+1}]} \zeta_s$, $\beta_i = \zeta_{u_i}$, $a = A(\alpha_1, \dots, \alpha_{n-1})$ and by convention $z_0 = 0$, $\alpha_0 = 0$.

We recall that, under $\boldsymbol{P}_0$, the law of $(\zeta_s;\, s \in [0, 1])$ is the law of normalized Brownian excursion. Thus, we are led to formulate the following proposition.

Proposition 3. *Let us fix $0 < u_1 < \cdots < u_n < 1$ and let $(\zeta_s)_{s \in [0,1]}$ have the law of a normalised Brownian excursion. Then the law of the $(2n-1)$-uple*

$$(\zeta_{u_1}, \dots, \zeta_{u_n}, \inf_{s \in [u_1, u_2]} \zeta_s, \dots, \inf_{s \in [u_{n-1}, u_n]} \zeta_s)$$

has a density with respect to the Lebesgue measure given by the formula

$$P\left[\zeta_{u_1} \in \mathrm{d}\beta_1, \ldots, \zeta_{u_n} \in \mathrm{d}\beta_n, \inf_{s\in[u_1,u_2]} \zeta_s \in \mathrm{d}\alpha_1, \ldots, \inf_{[u_{n-1},u_n]} \zeta_s \in \mathrm{d}\alpha_{n-1}\right]$$

$$= c(u_1, \ldots, u_n)\beta_1 \exp\left(-\frac{\beta_1^2}{2u_1}\right)$$

$$\times \prod_{i=1}^{n-1} 1_{\{0<\alpha_i<\beta_i\wedge\beta_{i+1}\}} (\beta_i + \beta_{i+1} - 2\alpha_i) \exp\left(-\frac{(\beta_i+\beta_{i+1}-2\alpha_i)^2}{2(u_{i+1}-u_i)}\right)$$

$$\times \beta_n \exp\left(-\frac{\beta_n^2}{2(1-u_n)}\right) \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \,\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1},$$

where $c(u_1, \ldots, u_n)$ is a positive constant.

Proof. We fix $0 = u_0 < u_1 < \cdots < u_n < u_{n+1} = 1$. We first suppose that $(B_s; s \in [0, 1])$ is, under P_{β_0}, a linear Brownian motion starting from $\beta_0 > 0$. A classical iterated application of Markov property and reflection principle gives

$$P_{\beta_0}\left[B_{u_1} \in \mathrm{d}\beta_1, \ldots, B_{u_n} \in \mathrm{d}\beta_n, B_1 \in \mathrm{d}\beta_{n+1}, \inf_{[0,u_1]} B \in \mathrm{d}\alpha_0, \ldots, \inf_{[u_n,1]} B \in \mathrm{d}\alpha_n\right]$$

$$= c(u_1, \ldots, u_n) \prod_{i=0}^{n} (\beta_i + \beta_{i+1} - 2\alpha_i)\mathrm{e}^{-(\beta_i+\beta_{i+1}-2\alpha_i)^2/2(u_{i+1}-u_i)} \quad \mathbf{1}_{\{0<\alpha_i<\beta_i\wedge\beta_{i+1}\}}$$

$$\mathrm{d}\beta_1 \ldots \mathrm{d}\beta_{n+1} \,\mathrm{d}\alpha_0 \ldots \mathrm{d}\alpha_n.$$

We condition the Brownian motion B to stay positive, that is $\alpha_i > 0$ for every $i \in \{0, \ldots, n\}$. Then we integrate with respect to α_0 and α_n, condition by the event $\{B_1 \in [0, \varepsilon]\}$ and let $\varepsilon\downarrow 0$. Finally, we let β_0 tend to 0 and get

$$\lim_{\beta_0\downarrow 0} P_{\beta_0}\left[B_{u_1} \in \mathrm{d}\beta_1, \ldots, B_{u_n} \in \mathrm{d}\beta_n, \inf_{[u_1,u_2]} B \in \mathrm{d}\alpha_1, \ldots, \inf_{[u_{n-1},u_n]} B \in \mathrm{d}\alpha_{n-1}\right.$$

$$\left.|\forall t \in [0, 1), B_t > 0; B_1 = 0\right]$$

$$= c(u_1, \ldots, u_n)\beta_1 \exp\left(-\frac{\beta_1^2}{2u_1}\right)$$

$$\times \prod_{i=1}^{n-1} 1_{\{0<\alpha_i<\beta_i\wedge\beta_{i+1}\}} (\beta_i + \beta_{i+1} - 2\alpha_i) \exp\left(-\frac{(\beta_i+\beta_{i+1}-2\alpha_i)^2}{2(u_{i+1}-u_i)}\right)$$

$$\times \beta_n \exp\left(-\frac{\beta_n^2}{2(1-u_n)}\right) \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \,\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1}.$$

If we integrate this density with respect to $\alpha_1, \ldots, \alpha_{n-1}$, we find the density of the finite-dimensional marginal laws of normalised Brownian excursion as given, for instance, in Blumenthal (1992, p. 42). So the law of Brownian motion, conditioned as specified in the previous formula, is the law of the normalised Brownian excursion. As a consequence the previous formula gives the sought-after density. □

3. Large deviations in the finite-dimensional case

This section is devoted to the proof of the following proposition.

Proposition 4. *Let $\sigma = [0 < u_1 < \cdots < u_n < 1]$ be a finite partition of* $[0, 1]$. *Under* $\boldsymbol{P}_0$, *the laws μ_ε of*

$$(\varepsilon \widehat{W}_{u_1}, \ldots, \varepsilon \widehat{W}_{u_n}, \varepsilon^{2/3}\zeta_{u_1}, \ldots, \varepsilon^{2/3}\zeta_{u_n},$$

$$\varepsilon^{2/3} \inf_{[u_1,u_2]} \zeta, \ldots, \varepsilon^{2/3} \inf_{[u_{n-1},u_n]} \zeta, \varepsilon \widehat{W}_{\operatorname{arginf}([u_1,u_2],\zeta)}, \ldots, \varepsilon \widehat{W}_{\operatorname{arginf}([u_{n-1},u_n],\zeta)})$$

satisfy a large deviation principle with speed $\varepsilon^{-4/3}$ and rate function

$$I_\sigma(y_1, \ldots, y_n, \beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_{n-1}, z_1, \ldots, z_{n-1})$$

$$= \frac{\beta_1^2}{2\, u_1} + \sum_{i=1}^{n-1} \frac{(\beta_i + \beta_{i+1} - 2\alpha_i)^2}{2(u_{i+1} - u_i)} + \frac{\beta_n^2}{2(1 - u_n)}$$

$$+ \sum_{i=1}^{n-1} \frac{|z_i - z_{a(i)}|^2}{2(\alpha_i - \alpha_{a(i)})} + \sum_{i=1}^{n} \frac{|y_i - z_{v_a(i)}|^2}{2(\beta_i - \alpha_{v_a(i)})}$$

if $0 < \alpha_i < \beta_i \wedge \beta_{i+1}$ for every i and $+\infty$ otherwise.

We start with two lemmas.

Lemma 5. *Let $(\mu_\varepsilon)_{\varepsilon\downarrow 0}$ be a family of probabilities on $\mathbb{R}^m$ and J be a good rate function (that is, J is a function on $\mathbb{R}^m$ with values in $[0, +\infty]$ such that $J^{-1}([0, L])$ is compact for every $L \in (0, +\infty)$). In order to show that $(\mu_\varepsilon)_{\varepsilon\downarrow 0}$ satisfies a large deviation principle with speed $1/\varepsilon$ and rate function J it suffices to show that, for every open subset B of $\mathbb{R}^m$,*

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \mu_\varepsilon(B) = -\inf_B J.$$

Proof. This lemma can be proved using Theorem 4.1.11 and Lemma 4.1.6 in Dembo and Zeitouni (1993b). For the convenience of the reader we give a short proof. We only have to show that, for every closed subset F of $\mathbb{R}^m$,

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mu_\varepsilon(F) \leqslant -\inf_F J.$$

For $\gamma > 0$, we define F_γ as the set of points in $\mathbb{R}^m$ whose distance to F is strictly smaller than γ. Since it is an open subset containing F we have

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mu_\varepsilon(F) \leqslant \limsup_{\varepsilon\downarrow 0} \varepsilon \log \mu_\varepsilon(F_\gamma) = -\inf_{F_\gamma} J.$$

To complete the proof of the lemma it suffices to check that $\lim_{\gamma\downarrow 0} \inf_{F_\gamma} J = \inf_F J$. If not there exists $\alpha < \inf_F J$ such that, for every $\gamma > 0$, $\inf_{F_\gamma} J \leqslant \alpha$. We choose $\alpha' \in (\alpha, \inf_F J)$ and, for every $\gamma > 0$, x_γ such that $J(x_\gamma) \leqslant \alpha'$. By compacity we may suppose that $x_\gamma \to x_0$. Then $x_0 \in F$ and $J(x_0) \leqslant \alpha' < \inf_F J$ which is absurd. □

Lemma 6. *Let B be an open subset of $\mathbb{R}^m$ and $F, G : B \to (0, +\infty]$ be continuous functions on B such that $\int_B G(x)\exp(-(F(x))\,\mathrm{d}x < +\infty$. Then*

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \int_B G(x) \exp -\left(\frac{1}{\varepsilon} F(x)\right) \mathrm{d}x = -\inf_B F.$$

Proof. This is an easy application of Laplace's method. □

Proof of Proposition 4. It is not hard to see that I_σ is good rate function. Then, Lemma 5 shows that it suffices to obtain that, for B open subset of $(\mathbb{R}^d)^n \times [0, +\infty)^n \times [0, +\infty)^{n-1} \times (\mathbb{R}^d)^{n-1}$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mu_\varepsilon(B) = -\inf_B I_\sigma. \tag{1}$$

We use Propositions 2 and 3 to get

$$\begin{aligned}\mu_\varepsilon(B) = \int_{B'} &\mathrm{d}y_1 \ldots \mathrm{d}y_n \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1} \mathrm{d}z_1 \ldots \mathrm{d}z_{n-1}\\
&\times c(u_1, \ldots, u_n)\beta_1\beta_n \prod_{i=1}^{n-1} \mathbf{1}_{\{0<\alpha_i<\beta_i\wedge\beta_{i+1}\}} (\beta_i + \beta_{i+1} - 2\alpha_i)\\
&\times \exp -\frac{1}{2}\left(\frac{\beta_1^2}{u_1} + \sum_{i=1}^{n-1} \frac{(\beta_i + \beta_{i+1} - 2\alpha_i)^2}{u_{i+1} - u_i} + \frac{\beta_n^2}{1 - u_n}\right)\\
&\times \prod_{i=1}^{n-1} p(\alpha_i - \alpha_{a(i)}; z_{a(i)}, z_i) \prod_{i=1}^{n} p(\beta_i - \alpha_{v_a(i)}; z_{v_a(i)}, y_i),\end{aligned}$$

where B' is the set of $y_1, \ldots, y_n, \beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_{n-1}, z_1, \ldots, z_{n-1}$ such that $(\varepsilon y_1, \ldots, \varepsilon y_n, \varepsilon^{2/3}\beta_1, \ldots, \varepsilon^{2/3}\beta_n, \varepsilon^{2/3}\alpha_1, \ldots, \varepsilon^{2/3}\alpha_{n-1}, \varepsilon z_1, \ldots, \varepsilon z_{n-1}) \in B$. We proceed to the changes of variables $\varepsilon y_i \to y_i$, $\varepsilon^{2/3}\beta_i \to \beta_i$, $\varepsilon^{2/3}\alpha_i \to \alpha_i$, $\varepsilon z_i \to z_i$ and reexpress the Gaussian transition densities:

$$\begin{aligned}\mu_\varepsilon(B) = c'(u_1, \ldots, u_n)\varepsilon^q \int_B &\mathrm{d}y_1 \ldots \mathrm{d}y_n \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1} \mathrm{d}z_1 \ldots \mathrm{d}z_{n-1}\\
&\times \beta_1\beta_n \prod_{i=1}^{n-1} (\beta_i + \beta_{i+1} - 2\alpha_i) \prod_{i=1}^{n-1} (\alpha_i - \alpha_{a(i)})^{-d/2} \prod_{i=1}^{n} (\beta_i - \alpha_{v_a(i)})^{-d/2}\\
&\times \exp -\frac{1}{\varepsilon^{4/3}} I_\sigma(y_1, \ldots, y_n, \beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_{n-1}, z_1, \ldots, z_{n-1}),\end{aligned}$$

where q denotes an integer constant. We deduce by Lemma 6 that Eq. (1) holds and Proposition 4 follows. □

4. A result of exponential tightness

Proposition 7. *The laws under P_0 of $((\varepsilon \hat{W}_s, \varepsilon^{2/3}\zeta_s); s \in [0, 1])$ as $\varepsilon\downarrow 0$ are exponentially tight with speed $\varepsilon^{-4/3}$.*

We state with two lemmas. The proof of the first one is omitted. We denote by $|\cdot|$ the usual euclidean norm on $\mathbb{R}^d$.

Lemma 8. *Let N be a centered Gaussian variable on $\mathbb{R}^d$ with variance $V\, I_d$. Then, for every $\lambda > 0$, we have the following inequality*:

$$E(\mathrm{e}^{\lambda|N|}) \leqslant 2d\, \mathrm{e}^{(d/2)\lambda^2 V}.$$

Lemma 9. *Let $(\zeta_t)_{t\in[0,1]}$ have the law of normalised Brownian excursion. Then there exist two positive constants c_1 and c_2 such that, for every $\alpha > 0$ and all $0 \leqslant s < t \leqslant 1$,*

$$E\left[\exp\left(\alpha \sup_{[u,v]\subset[s,t]} |\zeta_u - \zeta_v|\right)\right] \leqslant c_1\, \mathrm{e}^{c_2\alpha^2|t-s|}. \tag{2}$$

Proof. We first suppose that $0 \leqslant s < t \leqslant \frac{1}{2}$. We may write $\zeta_t = (1-t)|B_{t/1-t}|$ where B is a Brownian motion in $\mathbb{R}^3$ starting from 0 (cf. Blumenthal, 1992, p. 42). As a consequence, for $[u, v] \subset [s, t]$,

$$|\zeta_u - \zeta_v| \leqslant |B_{u/(1-u)} - B_{v/(1-v)}| + |t-s| \sup_{r\in[0,1]} |B_r|.$$

We use Cauchy–Schwarz inequality to treat separately the resulting two terms. A scaling argument for B gives the correct upper bound for the first term. The second term is easily treated.

Then, a time-reversal argument gives Eq. (2) in the case $1 \geqslant t > s \geqslant \frac{1}{2}$. Finally, in the case $0 \leqslant s \leqslant \frac{1}{2} \leqslant t \leqslant 1$, we write

$$\sup_{[u,v]\subset[s,t]} |\zeta_u - \zeta_v| \leqslant \sup_{[u,v]\subset[s,1/2]} |\zeta_u - \zeta_v| + \sup_{[u,v]\subset[1/2,t]} |\zeta_u - \zeta_v|,$$

and we use the Cauchy–Schwarz inequality again. □

We may now start the proof of Proposition 7. By Theorem 3 of [Schied (preprint)] it suffices to find a constant K such that, for all $0 \leqslant s < t \leqslant 1$,

$$\boldsymbol{P}_0\left[\exp\left(\frac{\varepsilon^{-4/3}}{|t-s|^{1/4}}(\varepsilon|\hat{W}_s - \hat{W}_t| + \varepsilon^{2/3}|\zeta_s - \zeta_t|)\right)\right] \leqslant K^{\varepsilon^{-4/3}}.$$

We first consider the conditional expectation with respect to (ζ_s). In this case $\hat{W}_s - \hat{W}_t$ has the law of a Brownian motion in $\mathbb{R}^d$ at time $\zeta_s + \zeta_t - 2\inf_{[s,t]}\zeta$. So by Lemma 8,

$$\boldsymbol{P}_0\left[\exp\left(\frac{\varepsilon^{-4/3}}{|t-s|^{1/4}}\,\varepsilon|\hat{W}_s - \hat{W}_t|\right)|(\zeta_s)\right] \leqslant 2d\exp\left(\frac{d\varepsilon^{-2/3}}{2|t-s|^{1/2}}\,|\zeta_s + \zeta_t - 2\inf_{[s,t]}\zeta|\right).$$

Obviously, $|\zeta_s + \zeta_t - 2\inf_{[s,t]}\zeta| \leqslant 2\sup_{[u,v]\subset[s,t]}|\zeta_u - \zeta_v|$. Finally, it suffices to show that

$$\boldsymbol{P}_0\left[\exp\left(\frac{d\varepsilon^{-2/3}}{|t-s|^{1/2}} \sup_{[u,v]\subset[s,t]} |\zeta_u - \zeta_v|\right)\right] \leqslant K^{\varepsilon^{-4/3}}.$$

But this is a consequence of Lemma 9 and the proof of Proposition 7 is complete. □

5. Proof of the main theorem

We want to show that Theorem 1 can be deduced from Proposition 4 and Proposition 7. To this end, we use Lemma 19 of Schied (1996). We denote by $\mathscr{S}$ the set of finite subdivisions of [0, 1] identified with the set of finite subsets of [0, 1] and equipped with the inclusion relation denoted by $\prec$. For $\sigma \in \mathscr{S}$, say σ is the subdivision $[0 < u_1 < \cdots < u_n < 1]$, we define a map

$$q_\sigma : \mathscr{E} \to (\mathbb{R}^d)^n \times [0, +\infty)^n \times [0, +\infty)^{n-1} \times (\mathbb{R}^d)^{n-1}$$

by the formula

$$q_\sigma(T, \eta) = (T_{u_1}, \dots, T_{u_n}, \eta_{u_1}, \dots, \eta_{u_n}, \inf_{[u_1, u_2]} \eta, \dots, \inf_{[u_{n-1}, u_n]} \eta, T_{m_1}, \dots, T_{m_{n-1}}).$$

Here $m_i \in [u_i, u_{i+1}]$ is such that η_{m_i} is the minimum of η on $[u_i, u_{i+1}]$. Such m_i is not necessarily unique but, since $(T, \eta) \in \mathscr{E}$ the value of T_{m_i} does not depend on the choice of m_i. For instance, we will take m_i as small as possible and denote this value by $m_i = \operatorname{arginf}([u_i, u_{i+1}], \eta)$.

We set $\mathscr{Y}_\sigma = \{q_\sigma(T, \eta); (T, \eta) \in \mathscr{E}\}$. For $\tau \prec \sigma$ we denote by $p_{\tau\sigma}$ the projection from $\mathscr{Y}_\sigma$ to $\mathscr{Y}_\tau$. With all these notations we can apply Lemma 19 of Schied (1996). It implies that the law of $((\varepsilon \hat{W}_s, \varepsilon^{2/3} \zeta_s); s \in [0, 1])$ under $\boldsymbol{P}_0$ satisfies a large deviation principle as $\varepsilon \downarrow 0$ with speed $\varepsilon^{-4/3}$ and good rate function

$$J(T, \eta) = \sup_{\sigma \in \mathscr{S}} I_\sigma(q_\sigma(T, \eta)). \tag{3}$$

Besides the supremum can also be taken over a sequence of refining subdivisions of [0, 1] with stepsize going to 0. The remaining problem is to show that this formula leads to the expression of J given in Theorem 1.

Lemma 10. *Let $[a, b]$ be a non-empty compact interval and f be an absolutely continuous function on $[a, b]$ with values in $\mathbb{R}^m$ $(m \geq 1)$. Let V be an open subset of $[a, b]$ and $V = \bigcup_{i \in I}]a_i, b_i[$ be its decomposition as union of its connected components. We suppose that, for $i \in I, f(a_i) = f(b_i)$ and we define a function g by setting $g(x) = f(x)$ if $x \notin V$ and $g(x) = f(a_i)$ if $x \in]a_i, b_i[$.*

Then g is absolutely continuous. For almost all $x \in [a, b]$, we have $\dot{g}(x) = 0$ if $x \in V$ and $\dot{g}(x) = \dot{f}(x)$ if $x \notin V$.

The proof is easy and omitted.

Proposition 11. *Let $\sigma = [0 < u_1 < \cdots < u_n < 1]$ be a partition of $[0, 1]$, $(U, \eta) \in \mathscr{HPW}$ and set, for every i, $m_i = \operatorname{arginf}([u_i, u_{i+1}], \eta)$, $\beta_i = \eta_{u_i}$, $\alpha_i = \eta_{m_i}$, $y_i = \hat{U}_{u_i}$, $z_i = \hat{U}_{m_i}$, $a = A(\alpha_1, \dots, \alpha_{n-1})$, $z_0 = 0$, $\alpha_0 = 0$. Then we have*

$$\frac{1}{2} \int_0^1 \frac{|\dot{\hat{U}}_s|^2}{|\dot{\eta}_s|} \mathrm{d}s \geq \sum_{i=1}^{n-1} \frac{|z_i - z_{a(i)}|^2}{\alpha_i - \alpha_{a(i)}} + \sum_{i=1}^{n} \frac{|y_i - z_{v_a(i)}|^2}{\beta_i - \alpha_{v_a(i)}}. \tag{4}$$

Proof. For every $i \in \{1, \dots, n-1\}$ we set $d_i = \inf\{s > m_i; \eta_s = \alpha_i\}$ and $g_i = \sup\{s < m_i; \eta_s = \alpha_i\}$ (with the convention $\inf \emptyset = \sup \emptyset = m_i$). We also set $m_0 = 0$ and $d_0 = 1$; see Fig. 2.

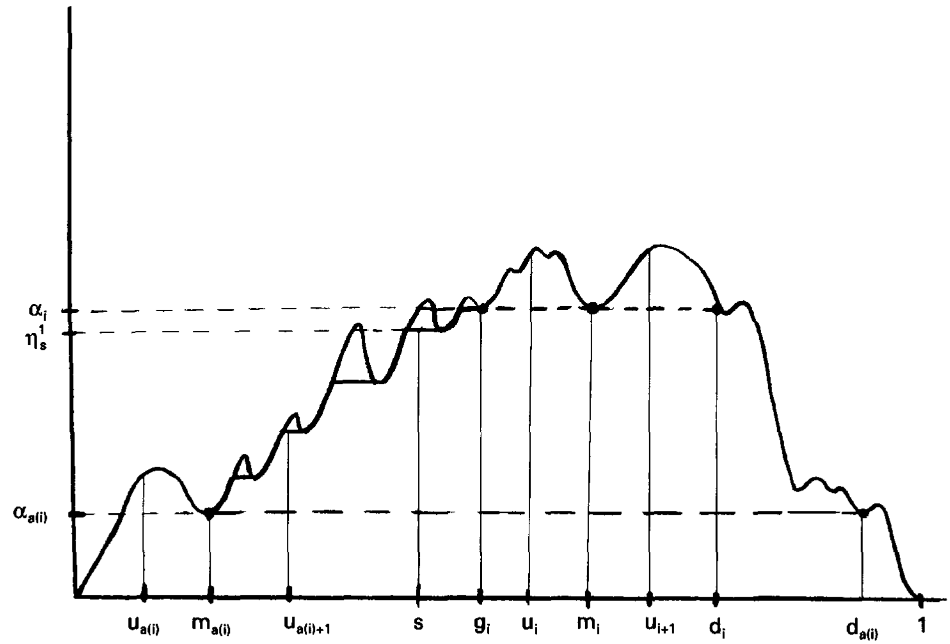

Fig. 2.

We now fix $i \in \{1, \dots, n-1\}$ and suppose for instance that $a(i) < i$ so that $[g_i, d_i] \subset (m_{a(i)}, d_{a(i)})$. We are going to show that

$$\int_{G_i} \frac{|\dot{\hat{U}}_s|^2}{|\dot{\eta}_s|} \, \mathrm{d}s \geqslant \frac{|z_i - z_{a(i)}|^2}{\alpha_i - \alpha_{a(i)}}, \tag{5}$$

where $G_i = [m_{a(i)}, g_i)$. We recall that $\eta_{m_{a(i)}} = \alpha_{a(i)} < \alpha_i = \eta_{g_i}$. We set for $s \in [m_{a(i)}, g_i]$, $\eta_s^1 = \inf\{\eta_r;\ r \in [s, g_i]\}$ and define a path-valued process by the formula $U_s^1(u) = U_s(u \wedge \eta_s^1)$. In other words, the new lifetime η^1 is obtained by making the ancient lifetime η constant during the excursions over its future infimum. On such an excursion, η_s^1 remains constant, all the paths $U_s(\cdot)$ coincide up to time η_s^1 and $\hat{U}_s^1$ is constant. Since we are in the situation of Lemma 10 we claim that $\eta^1, \hat{U}^1$ are absolutely continuous. Moreover, Lemma 10 also proves that the integral $\int_{m_{a(i)}}^{g_i} (|\dot{\hat{U}}_s^1|^2/|\dot{\eta}_s^1|) \, \mathrm{d}s$ (with the convention $0/0 = 0$) is smaller than $\int_{m_{a(i)}}^{g_i} (|\dot{\hat{U}}_s|^2/|\dot{\eta}_s|) \, \mathrm{d}s$ since in the latter integral we simply replace the non-negative integrated function by 0 on a certain set. We now apply the Cauchy–Schwarz inequality

$$\left(\int_{m_{a(i)}}^{g_i} |\dot{\hat{U}}_s^1| \, \mathrm{d}s\right)^2 \leqslant \left(\int_{m_{a(i)}}^{g_i} \frac{|\dot{\hat{U}}_s^1|^2}{|\dot{\eta}_s^1|} \, \mathrm{d}s\right)\left(\int_{m_{a(i)}}^{g_i} |\dot{\eta}_s^1| \, \mathrm{d}s\right).$$

The left-hand side of this inequality is greater than

$$|\hat{U}_{m_{a(i)}}^1 - \hat{U}_{g_i}^1|^2 = |\hat{U}_{m_{a(i)}} - \hat{U}_{g_i}|^2 = |z_{a(i)} - z_i|^2.$$

Since η^1 is monotone the integral $\int_{m_{a(i)}}^{g_i} |\dot{\eta}_s^1| \, \mathrm{d}s$ may be computed without the absolute values and its value is $\eta_{g_i}^1 - \eta_{m_{a(i)}}^1 = \alpha_i - \alpha_{a(i)}$. Finally, we get Eq. (5) as desired. By a similar method we obtain that Eq. (5) holds if G_i is replaced by $G_i' = [d_i, d_{a(i)})$. In the case $i < a(i)$ we obtain the same result with $G_i = [g_{a(i)}, g_i)$, $G_i' = [d_i, m_{a(i)})$.

By similar arguments we obtain, for $i \in \{1, \dots, n\}$,

$$\int_{H_i \cup H_i'} \frac{|\dot{\hat{U}}_s|^2}{|\dot{\eta}_s|} \mathrm{d}s \geqslant 2 \frac{|y_i - z_{v_a(i)}|^2}{\beta_i - \alpha_{v_a(i)}} \tag{6}$$

with $H_i = [g_i, u_i)$, $H_i' = [u_i, m_i)$ if $v(i) = i$ and $H_i = [m_{i-1}, u_i)$, $H_i' = [u_i, d_{i-1})$ if $v(i) = i - 1$.

Finally, it is easy to check that the sets G_i, G_i', $i \in \{1, \dots, n-1\}$ and H_i, H_i', $i \in \{1, \dots, n\}$ are disjoint and their union is equal $[0, 1)$. We complete the proof by combining Eqs. (5) and (6). □

Remark. We note that in (4) the equality is realised in particular if we take a process $(U^\sigma, \eta^\sigma) \in \mathscr{H}\mathscr{P}\mathscr{W}$ such that $\dot{\eta}_s^\sigma$ and $\dot{U}_s^\sigma$ are constant over the sets G_i, G_i', H_i, H_i'.

Lemma 12. *The continuous function f: $[0, 1] \to \mathbb{R}$ is absolutely continuous on $[0, 1]$ with square integrable derivative (we write $f \in H^1([0, 1])$) if and only if*

$$M(f) = \sup \sum_{i=1}^{n-1} \frac{(f(u_i) + f(u_{i+1}) - 2 \inf_{[u_i, u_{i+1}]} f)^2}{u_{i+1} - u_i} < +\infty,$$

where the supremum is taken over all the finite partitions $[0 < u_1 < \cdots < u_n < 1]$ of $[0, 1]$. In this case

$$M(f) = \int_0^1 \dot{f}^2(s) \, \mathrm{d}s.$$

Proof. We know that the same lemma holds if $M(f)$ is replaced by

$$M'(f) = \sup \sum_{i=1}^{n-1} \frac{(f(u_{i+1}) - f(u_i))^2}{u_{i+1} - u_i}$$

as is it proved, for instance, in Revuz and Yor (1994, p. 328). But we have

$$|f(u_{i+1}) - f(u_i)| \leqslant f(u_i) + f(u_{i+1}) - 2 \inf_{[u_i, u_{i+1}]} f$$

so that $M'(f) \leqslant M(f)$. Conversely, we introduce the reals m_i such that $f(m_i) = \inf_{[u_i, u_{i+1}]} f$ and remark by the convexity of the square function that

$$\frac{(f(u_i) + f(u_{i+1}) - 2 \inf_{[u_i, u_{i+1}]} f)^2}{u_{i+1} - u_1} \leqslant \frac{(f(m_i) - f(u_i))^2}{m_i - u_i} + \frac{(f(u_{i+1}) - f(m_i))^2}{u_{i+1} - m_i}.$$

Hence, the consideration of the subdivision $[0 < u_1 < m_1 < u_2 < \cdots < m_{n-1} < u_n]$ gives $M(f) \leqslant M'(f)$ and we conclude $M(f) = M'(f)$. □

Lemma 13. *Let (T, η) be such that $\eta \in H^1([0, 1]$ and T: $[0, 1] \to \mathbb{R}^d$ is absolutely continuous. Let $[0 < u_1^p < \cdots < u_{n_p}^p < 1]$ be a sequence of subdivisions with stepsize* *going to* 0 *as* $p \to +\infty$. *Then we have*

$$\liminf_{p\to+\infty} \sum_{i=1}^{n_p-1} \frac{|T(u^p_{i+1}) - T(u^p_i)|^2}{|\eta(u^p_{i+1}) - \eta(u^p_i)|} \geqslant \int_0^1 \frac{|\dot{T}(s)|^2}{|\dot{\eta}(s)|}\,ds.$$

Proof. This is a consequence of Fatou's lemma. □

We may now complete the proof of Theorem 1. We fix $(T,\eta)\in\mathscr{E}$. By the definition of $\mathscr{E}$ we may find $(U,\eta)\in\mathscr{PW}$ such that $(T,\eta)=(\hat{U},\eta)$. For every $\sigma\in\mathscr{S}$ we denote by $G^\sigma_i, G'^\sigma_i, H^\sigma_i, H'^\sigma_i$ the sets introduced in the proof of Proposition 11. We also denote $(U^\sigma,\eta^\sigma)\in\mathscr{HPW}$ the piecewise linear process such that $(\hat{U}^\sigma,\eta^\sigma)\in\mathscr{HE}$ coincides with (T,η) at the boundaries of the sets $G^\sigma_i, G'^\sigma_i, H^\sigma_i, H'^\sigma_i$. As a consequence we have $q_\sigma(\hat{U}^\sigma,\eta^\sigma)=q_\sigma(\hat{U},\eta)$ and as it is noticed in the remark following the proof of Proposition 11, the equality holds in (4). Thus, Eq. (3) giving the rate function J may be written as

$$J(T,\eta) = \sup_{\sigma\in\mathscr{S}} \frac{1}{2}\left\{\frac{\eta^2_{u_1}}{u_1} + \sum_{i=1}^{n-1} \frac{(\eta_{u_i}+\eta_{u_{i+1}} - 2\inf_{[u_i,u_{i+1}]}\eta)^2}{u_{i+1}-u_i} + \frac{\eta^2_{u_n}}{1-u_n} + \frac{1}{2}\int_0^1 \frac{|\dot{\hat{U}}^\sigma_s|^2}{|\dot{\eta}^\sigma_s|}\,ds\right\}. \tag{7}$$

We first suppose that $J(T,\eta)$ is finite and show that necessarily $(T,\eta)\in\mathscr{HE}$. First Lemma 12 implies that $\eta\in H^1([0,1])$. We now pass to the absolute continuity of T. We take $0<a_1<b_1<a_2<\cdots<a_n<b_n<1$. We consider $(U^{\tilde\sigma},\eta^{\tilde\sigma})\in\mathscr{HPW}$ associated to the subdivision $\tilde\sigma=[0<a_1<b_1<a_2<\cdots<a_n<b_n<1]$ in the way described previously, in particular such that $(\hat{U}^{\tilde\sigma},\eta^{\tilde\sigma})$ coincides with (T,η) at all the points $0,a_1,b_1,a_2,\ldots,a_n,b_n,1$. We remark that any of the intervals $G^{\tilde\sigma}_i, G'^{\tilde\sigma}_i, H^{\tilde\sigma}_i, H'^{\tilde\sigma}_i$ is included in one of the intervals determined by the subdivision $[0<a_1<b_1<a_2<\cdots<a_n<b_n<1]$. We denote by S the set of all the intervals $G^{\tilde\sigma}_i, G'^{\tilde\sigma}_i, H^{\tilde\sigma}_i, H'^{\tilde\sigma}_i$ included in one of the intervals $[a_i,b_i]$. We note that each interval $[a_i,b_i]$ is the union of intervals belonging to S. Then by the Cauchy–Schwarz inequality and with obvious notations

$$\sum_{j=1}^n |T_{b_j}-T_{a_j}| \leqslant \sum_S |\Delta T| \leqslant \left(\sum_S \frac{|\Delta T|^2}{|\Delta\eta|}\right)^{1/2}\left(\sum_S |\Delta\eta|\right)^{1/2}.$$

But, on the one hand,

$$\sum_S \frac{|\Delta T|^2}{|\Delta\eta|} \leqslant \sum_{G_i,G'_i,H_i,H'_i} \frac{|\Delta T|^2}{|\Delta\eta|} = \int_0^1 \frac{|\dot{\hat{U}}^{\tilde\sigma}_s|^2}{|\dot{\eta}^{\tilde\sigma}_s|}\,ds \leqslant 4J(T,\eta) < +\infty$$

and, on the other hand,

$$\sum_S |\Delta\eta| \leqslant \int_{\bigcup_{i=1}^n [a_i,b_i]} |\dot{\eta}_s|\,ds.$$

It easily follows that T is absolutely continuous.

From now on we make the assumption that $(T,\eta)\in\mathscr{HE}$. We consider a sequence $\sigma_p=[0<u^p_1<\cdots<u^p_{n_p}<1]$ of refining subdivisions of $[0,1]$ with stepsize going to 0. Lemma 13 tells us that

$$\int_0^1 \frac{|\dot{T}_s|^2}{|\dot{\eta}_s|} \mathrm{d}s \leqslant \liminf_{p\to+\infty} \int_0^1 \frac{|\dot{\hat{U}}_s^{\sigma_p}|^2}{|\dot{\eta}_s^{\sigma_p}|} \mathrm{d}s.$$

But Proposition 11 shows that for every p the integral on the left-hand side is greater than the integral on the right-hand side. The previous inequality is thus an equality and the lim inf is in fact a limit which is reached from below. Now we pass to the limit in Eq. (7) using the previous result and Lemma 12. We get

$$J(T,\eta) = \frac{1}{2}\int_0^1 \dot{\eta}_s^2 \,\mathrm{d}s + \frac{1}{4}\int_0^1 \frac{|\dot{T}_s|^2}{|\dot{\eta}_s|} \mathrm{d}s,$$

as desired. □.

6. Applications

We start with an estimate of the probability that the Brownian snake exits a "big" ball. We denote by $\mathscr{B}(0,R)$ the open ball in $\mathbb{R}^d$ with centre 0 and radius R, by $\bar{\mathscr{B}}(0,R)$ its closure and by $\mathscr{B}(0,R)^c$ its complement.

Proposition 14. *Let us denote by $\mathscr{R}$ the range of the Brownian snake: $\mathscr{R} = \{\hat{W}_s;\ s \in [0,1]\}$. Then*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \boldsymbol{P}_0 \left[\mathscr{R} \cap \mathscr{B}\left(0,\frac{1}{\varepsilon}\right)^c \neq \emptyset\right] = -\frac{3}{2}.$$

Proof. We define A (resp. A') as the set of $(U,\eta) \in \mathscr{H}\mathscr{P}\mathscr{W}$ such that $\{\hat{U}_s;\ s\in[0,1]\}$ intersects $\mathscr{B}(0,1)^c$ (resp. $\bar{\mathscr{B}}(0,1)^c$). The set A is closed and A' is open. An application of Theorem 1 shows that it suffices to prove that

$$\inf_{(U,\eta)\in A} J(\hat{U},\eta) = \inf_{(U,\eta)\in A'} J(\hat{U},\eta) = \tfrac{3}{2}.$$

We prove the equality $\inf_A J = \frac{3}{2}$ (the equality $\inf_{A'} J = \frac{3}{2}$ is then proved by similar arguments). We consider $(U,\eta)\in A$ and call u the instant such that $|\hat{U}_u| \geqslant 1$. Proposition 11 shows that

$$\int_0^1 \frac{|\dot{\hat{U}}_s|^2}{|\dot{\eta}_s|} \mathrm{d}s \geqslant 2\,\frac{|\hat{U}_u|^2}{\eta_u} \geqslant \frac{2}{\eta_u}.$$

An easy application of Cauchy–Schwarz inequality leads to

$$\int_0^1 |\dot{\eta}_s|^2 \,\mathrm{d}s \geqslant \int_0^u |\dot{\eta}_s|^2 \,\mathrm{d}s + \int_u^1 |\dot{\eta}_s|^2 \,\mathrm{d}s \geqslant \frac{\eta_u^2}{u} + \frac{\eta_u^2}{1-u}.$$

As a consequence, if we denote by β the value η_u we get

$$J(\hat{U},\eta) \geqslant \frac{1}{2\beta} + \frac{\beta^2}{2u} + \frac{\beta^2}{2(1-u)}.$$

Moreover, the equality holds for a certain piecewise linear (U, η). It follows that

$$\inf_A J = \inf\left\{\frac{1}{2\beta}+\frac{\beta^2}{2u}+\frac{\beta^2}{2(1-u)};\ u\in[0,1],\ \beta>0\right\}=\frac{3}{2}. \qquad \square$$

We shall now give a new proof of the following result, which is due to Dembo and Zeitouni (1993). Note however that Dembo and Zeitouni obtain a different multiplicative constant ($3/2^{-5/3}$ instead of $\frac{3}{2}$). This is so because they deal with a process (ζ_s) distributed as twice the normalised Brownian excursion, as in Aldous (1993b).

We consider n disjoint balls $\mathscr{B}(x_1,\delta),\dots,\mathscr{B}(x_n,\delta)$ in $\mathbb{R}^d$ of radius $\delta>0$. As in Dembo and Zeitouni (1993a), we denote by $ST^\delta(\underline{x})$ the infimum of the sum of edge length over all embedded binary trees in $\mathbb{R}^d$ which intersect each of the balls $\mathscr{B}(x_1,\delta),\dots,\mathscr{B}(x_n,\delta)$.

Theorem 15.

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{4/3}\log \mathbf{P}_0[\forall i:\varepsilon\mathscr{R}\cap\mathscr{B}(x_i,\delta)\neq\emptyset] = -\tfrac{3}{2}\,(\mathrm{ST}^\delta(\underline{x}))^{4/3}.$$

Proof. As in the proof of the previous proposition the problem is to show that

$$\inf_{(U,\eta)\in A} J(\hat U,\eta)=\tfrac{3}{2}\,(\mathrm{ST}^\delta(\underline{x}))^{4/3}$$

where A is the set of $(U,\eta)\in\mathscr{HPW}$ such that $\{\hat U_s;\ s\in[0,1]\}$ intersects $\mathscr{B}(x_i,\delta)$ for every $i\in\{1,\dots,n\}$. We take $(U,\eta)\in A$ and choose $u_1,\dots,u_n$ such that $\hat U_{u_i}\in\mathscr{B}(x_i,\delta)$ for every i. We use the notations of Proposition 11 and we also call $l_1,\dots,l_{2n-1}$ (resp. $h_1,\dots,h_{2n-1}$) the quantities $|z_i-z_{a(i)}|$ $(i\in\{1,\dots,n-1\})$, $|y_i-z_{v_a(i)}|$ $(i\in\{1,\dots,n\})$ (resp. $\alpha_i-\alpha_{a(i)}$ $(i\in\{1,\dots,n-1\})$, $\beta_i-\alpha_{v_a(i)}$ $(i\in\{1,\dots,n\})$). We also denote by $t_1,\dots,t_{2n-1}$ (resp. $t'_1,\dots,t'_{2n-1}$) the lengths of the intervals G_i $(i\in\{1,\dots,n-1\})$, H_i $(i\in\{1,\dots,n\})$ (resp. G'_i $(i\in\{1,\dots,n-1\})$, H'_i $(i\in\{1,\dots,n\})$). We decompose the interval $[0,1]$ as the union of the intervals G_i, G'_i, H_i, H'_i and apply on each of these intervals the Cauchy–Schwarz inequality to get

$$\int_0^1 |\dot\eta_s|^2\,\mathrm{d}s \geqslant \sum_{i=1}^{2n-1} h_i^2\left(\frac{1}{t_i}+\frac{1}{t'_i}\right).$$

Then Proposition 11 tells us that

$$\frac{1}{2}\int_0^1 \frac{|\dot{\hat U}_s|^2}{|\eta_s|} \geqslant \sum_{i=1}^{2n-1}\frac{l_i^2}{h_i}.$$

Moreover, we know that the two previous inequalities turn to equalities for a certain choice of piecewise linear $(\hat U,\eta)$. As a consequence we claim that

$$\inf_{(U,\eta)\in A} J(\hat U,\eta)=\inf\left\{\frac{1}{2}\sum_{i=1}^{2n-1}h_i^2\left(\frac{1}{t_i}+\frac{1}{t'_i}\right)+\frac{1}{2}\sum_{i=1}^{2n-1}\frac{l_i^2}{h_i}\right\}$$

with the constraints $h_i>0$ for every i, $\sum_i l_i\geqslant \mathrm{ST}^\delta(\underline{x})$ and $\sum_i(t_i+t'_i)=1$. To perform this computation we first minimise over t_i, t'_i under the constraint $t_i+t'_i=T_i$ (fixed) which gives $t_i=t'_i=T_i/2$. Then we minimise over $h_i>0$ which leads to the value $h_i=2^{-1}l_i^{2/3}T_i^{1/3}$. Thus, we have in fact to minimise $\sum_i 3l_i^{4/3}/(2T_i^{1/3})$ under the constraints $\sum_i l_i \geqslant \mathrm{ST}^\delta(\underline{x})$ and $\sum_i T_i = 1$. The first-order condition on the Lagrangian proves that the minimum is reached for T_i proportional to l_i hence $T_i = l_i/\sum_i l_i$ and we get the desired infimum. □

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